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Module - V

Vortex filament and ~~Biort~~ Biot-Savart law :-

⇒ A Schematic of the flow induced by a point vortex of strength Γ located at a given point 'O'.

⇒ A straight line $\perp$er to the page, going through the point O, and extending to infinity both out of and into the page. This line is a straight vortex filament of strength Γ.

⇒ The flow induced in any plane $\perp$er to the straight vortex filament by the filament itself is identical to that induced by a point vortex of strength Γ; that is, the flow in the planes $\perp$er to the vortex filament at O (and) O' are identical to each other and are identical to the flow induced by a point vortex of strength Γ.

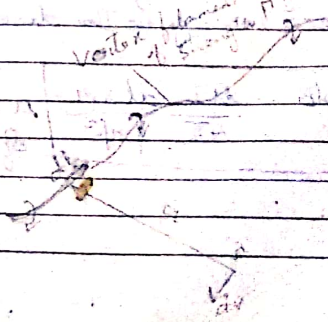
⇒ A straight vortex filament extending to $\pm\infty$

⇒ In general, a vortex filament can be curved.

⇒ The filament induces a flow field in the surrounding space.

⇒ If the circulation is taken about any path enclosing the filament, a constant value of Γ is obtained.

⇒ Hence the strength of vortex filament is defined as Γ.



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⇒ Consider a directed segment of the filament dl. The radius vector from dl to an arbitrary point 'P' in space is 'r'. The segment dl induces a velocity at P equal to

$$\boxed{dv = \frac{\Gamma}{4\pi} \frac{dl \times r}{|r|^3}}$$

This equation is called the Biot-Savart law and it has an analogy with electromagnetic theory.

⇒ If the vortex filament were instead visualized as a wire carrying an electric current I, then the magnetic field strength dB induced at point P by a segment of the wire dl with the current moving in the direction of dl is

$$\boxed{dB = \frac{\mu I}{4\pi} \frac{dl \times r}{|r|^3}}$$ μ = permeability of medium.

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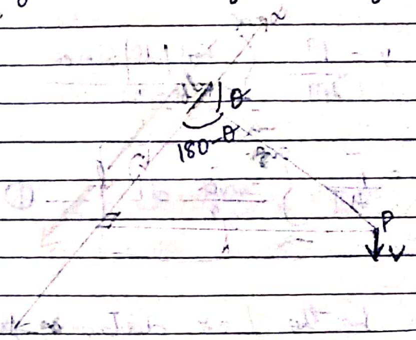
⇒ vortex filament and the associated Biot-Savart law are simply conceptual aerodynamic tools to be used for synthesizing more complex flows of an inviscid incompressible fluids.

Case I :- Apply Biot-Savart law to a straight vortex filament of infinite length.



The strength of the filament is Γ. The velocity induced at P by the entire vortex filament is

$$V = \int_{-\infty}^{\infty} \frac{\Gamma}{4\pi} \frac{dl \times r}{|r|^3}$$

$$V = \frac{\Gamma}{4\pi}\int_{-\alpha}^{\alpha}\frac{dl \times r}{|r|^3}$$

We know that the vector product (Cross product) of 2 vectors A & B is defined as

$$A \times B = |A||B|\sin\theta$$

$$V = \frac{\Gamma}{4\pi}\int_{-\alpha}^{\alpha}\frac{|r||dl|\sin\theta}{|r|^3}$$

$$\boxed{V = \frac{\Gamma}{4\pi}\int_{-\alpha}^{\alpha}\frac{\sin\theta}{r^2}\,dl} \quad \text{— ①}$$

Let 'h' be the ⊥er distance from point p to the Vortex filament. From geometry

$\sin(180-\theta) = \sin\theta$

$$\sin\theta = \frac{h}{r}$$

$$\boxed{r = \frac{h}{\sin\theta}}$$

$$-\tan\theta = \frac{h}{l}$$

$$\therefore \boxed{l = \frac{h}{-\tan\theta}}$$

On differentiating

$$\boxed{dl = \frac{-h}{\sin^2\theta}\,d\theta}$$

Substitute values in Eqn ①

$$V = \frac{\Gamma}{4\pi}\int_{-\alpha}^{\alpha}\frac{\sin\theta}{r^2}\times\frac{-h}{\sin^2\theta}\,d\theta$$

$$= \frac{\Gamma}{4\pi}\int_{-\alpha}^{\alpha}\frac{\sin\theta}{\left(\frac{h^2}{\sin^2\theta}\right)}\times\frac{-h}{\sin^2\theta}\,d\theta$$

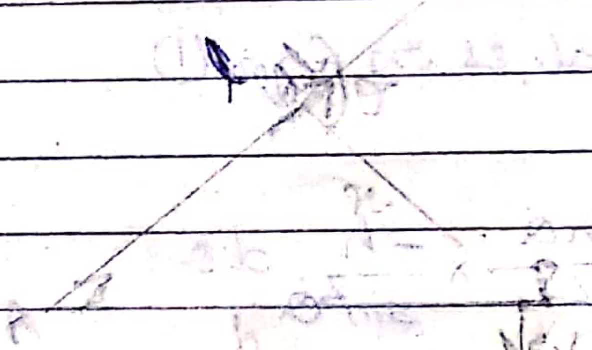
$$= \frac{-\Gamma}{4\pi h}\int_{\pi}^{0}\sin\theta\,d\theta$$

$$= \frac{-\Gamma}{4\pi h}\left[-\cos\theta\right]_{\pi}^{0}$$

$$= \frac{\Gamma}{4\pi h}\times[1 - -1] = \frac{\Gamma}{2\pi h}$$

Thus the velocity induced at a given point P by an infinite, straight vortex filament at a ⊥er distance h from P is $\dfrac{\Gamma}{2\pi h}$

Case 2 :- Apply Biot-Savart law on Semi infinite vortex filament



The filament extends from A to α.
Point A can be considered a boundary of the flow.
Let P be a point in the plane through 'A' ⊥er to the filament
Then by an integration similar to that of infinite vortex filament, the velocity induced at P by semi-infinite vortex filament is

$$V = \frac{-\Gamma}{4\pi h}\int_{\pi/2}^{0} \sin\theta \, d\theta$$

$$= \boxed{V = \frac{\Gamma}{4\pi h}}$$

* Helmholtz vortex theorem:

→ The strength of the vortex filament (Γ) is constant along its length.

→ The vortex element cannot end in a ~~field~~ fluid, it must extend to boundaries of the fluid (which can be $\pm\infty$) or formal closed path.

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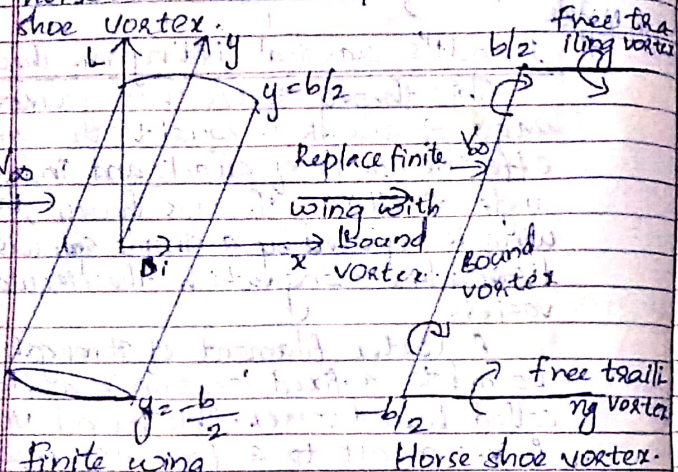
* Prandtl's classical lifting line theory:

This theory states that geometric angle of attack is equal to the sum of effective angle of attack and induced angle of attack. In this theory, the wing is replaced by a simple spanwise lifting line along which the circulation varies.

A vortex filament of strength 'Γ' is bound to a fixed location in a flow is called bound vortex. This bound vortex is in contrast to a free vortex which moves with the same fluid elements throughout the flow. Let us replace a finite wing of span 'b' with a bound vortex extending from $y = -\frac{b}{2}$ to $y = \frac{b}{2}$.

Due to Helmholtz theorem, a vortex filament cannot end in the fluid. Assume the vortex filament continues as 2 free vortices trailing downstream from the wing tips to infinity. This vortex (the bound plus 2 free) is in the shape of a horseshoe and therefore is called a horse shoe vortex.

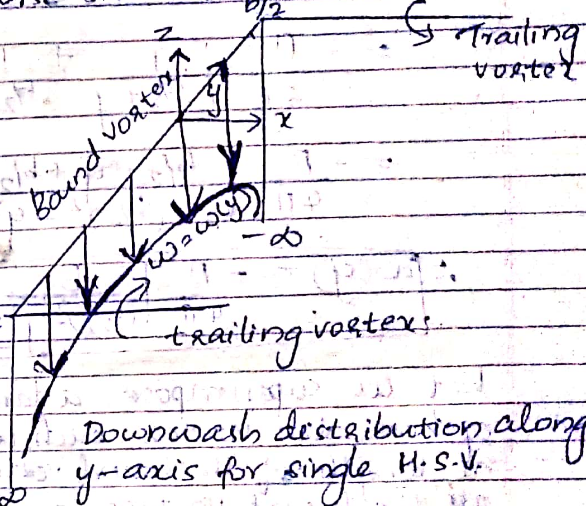


Finite wing

Horse shoe vortex.

A system of vortices trails downstream from the lifting line, which induces a downwash at the lifting line.

Consider the downwash 'w' induced along the bound vortex from $-\frac{b}{2}$ to $\frac{b}{2}$ by the horse shoe vortex:



Downwash distribution along y-axis for single H.S.V.

The velocity at any point y along the bound vortex induced by the trailing semi-infinite vortices is:

$$\boxed{V = \frac{\Gamma}{4\pi h}} \Rightarrow \text{General Eqn.}$$

Here

$$\boxed{w(y) = \frac{-\Gamma}{4\pi\left(\frac{b}{2}+y\right)} - \frac{\Gamma}{4\pi\left(\frac{b}{2}-y\right)}}$$

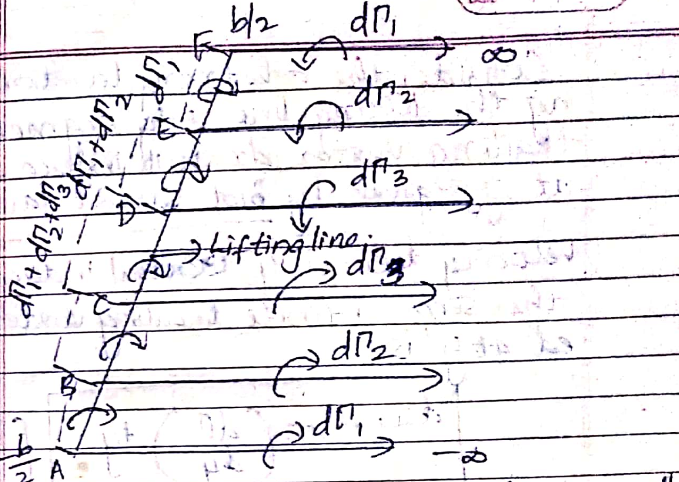
$$= \frac{-\Gamma}{4\pi}\left[\frac{1}{b/2+y} + \frac{1}{b/2-y}\right]$$

$$= \frac{-\Gamma}{4\pi}\left[\frac{b/2 - y + b/2 + y}{(b/2)^2 - y^2}\right]$$

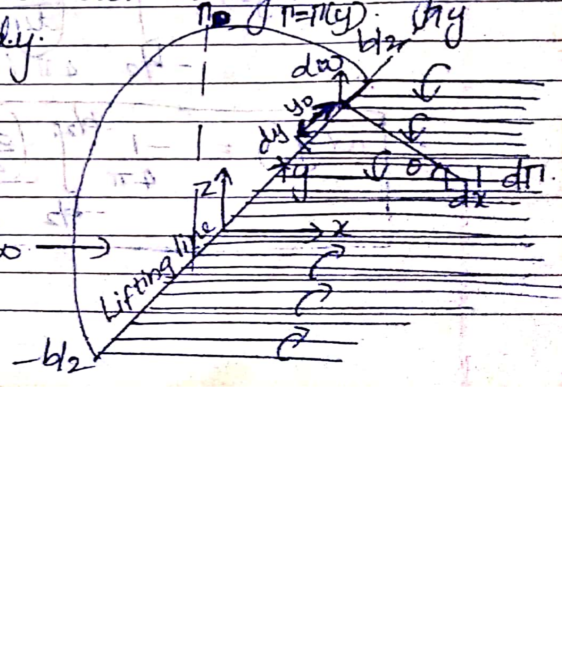
$$\therefore \boxed{w(y) = \frac{-\Gamma}{4\pi}\left[\frac{b}{(b/2)^2 - y^2}\right]}$$

Let us superimpose a large number of horse shoe vortex each with different length of the bound vortex, but with all the bound vortices are coincident along a single line called the lifting line.

Figure:



Let us single out an infinitesimally small segment of the lifting line 'dy' located at the coordinate 'y'. The circulation at y is $\Gamma(y)$ & the change in circulation over the segment dy is $d\Gamma = \left(\frac{d\Gamma}{dy}\right) dy$.



Consider the arbitrary location y_0 along the lifting line. Any segment of trailing vortex dx will induce a velocity at y_0 given by Biot-Savart law. The velocity dw at y_0 ~~located~~ induced by the semi-infinite trailing vortex located at y is:

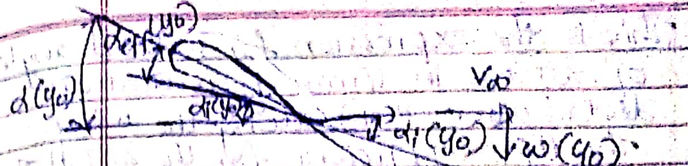
$$\boxed{dw = -\frac{\left(\frac{d\Gamma}{dy}\right)dy}{4\pi(y_0-y)}}$$

[Γ is decreasing, so -ve sign]

Total velocity w induced at y_0 is:

$$w(y_0) = \int_{-b/2}^{b/2} \frac{-\left(\frac{d\Gamma}{dy}\right)dy}{4\pi(y_0-y)}$$

$$\therefore \boxed{w(y_0) = \frac{-1}{4\pi}\int_{-b/2}^{b/2} \frac{\left(\frac{d\Gamma}{dy}\right)dy}{y_0-y}} \rightarrow ①$$



$$\boxed{\alpha_{eff}(y_0) = \alpha(y_0) - \alpha_i(y_0)} \rightarrow \text{I}$$

The induced angle of attack

$$\alpha_i(y_0) = \tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right)$$

α_i is small. So $\tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right) = \frac{-w(y_0)}{V_\infty}$

$$\therefore \boxed{\alpha_i(y_0) = \frac{-w(y_0)}{V_\infty}}$$

Substitute the value of $w(y_0)$

$$\therefore \alpha_i(y_0) = \frac{\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{\left(\frac{d\Gamma}{dy}\right)dy}{y_0-y}}{V_\infty}$$

$$\therefore \boxed{\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{\left(\frac{d\Gamma}{dy}\right)dy}{y_0-y}} \rightarrow ②$$

This is the expression for Induced angle of attack in terms of circulation distribution $\Gamma(y)$ along the wing.

The lift coefficient for the airfoil at $y = y_0$ is:

$$\boxed{C_l = a_0\left(\alpha_{eff}(y_0) - \alpha_{L=0}(y_0)\right)}$$

$$\boxed{C_l = 2\pi\left(\alpha_{eff}(y_0) - \alpha_{L=0}(y_0)\right)} \rightarrow ③$$

From Kutta-Joukowski theorem:

$$\boxed{L' = \rho_\infty V_\infty \Gamma(y_0)} \rightarrow ⓐ$$

$$\boxed{L' = \frac{1}{2}\rho_\infty V_\infty^2 c(y_0) C_l} \rightarrow ⓑ$$

ⓐ = ⓑ

$$\frac{1}{2}\rho_\infty V_\infty^2 c(y_0) C_l = \rho_\infty V_\infty \Gamma(y_0)$$

$$\boxed{C_l = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}}$$

Substitute this value in eqn ③:

$$\frac{2\Gamma(y_0)}{V_\infty c(y_0)} = 2\pi\left(\alpha_{eff}(y_0) - \alpha_{L=0}(y_0)\right)$$

$$\therefore \boxed{\alpha_{eff}(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0)} \rightarrow ⑥$$

From eqn Ⓘ, we know that:

$$\boxed{\alpha_{eff}(y_0) = \alpha(y_0) - \alpha_i(y_0)}$$

Substitute the value of $\alpha_i(y_0)$ & $\alpha_{eff}(y_0)$

$$\therefore \boxed{\alpha(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{\left(\frac{d\Gamma}{dy}\right)dy}{y_0 - y} + \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0)}$$

This is the fundamental equation of prandtl's lifting line theory. It states that geometric angle of attack is equal to the sum of effective angle of attack plus the induced angle of attack.

i) The lift distribution is obtained from the kutta-Joukowski theorem:

$$L'(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

ii) Total lift over the span:

$$L = \int_{-b/2}^{b/2} L'(y)\,dy$$

$$= \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,dy \qquad \text{where } L = \frac{1}{2}\rho_\infty V_\infty^2 S C_l$$

i.e., $$\frac{1}{2}\rho_\infty V_\infty^2 S C_l = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,dy$$

$$\therefore \boxed{C_l = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\,dy}$$

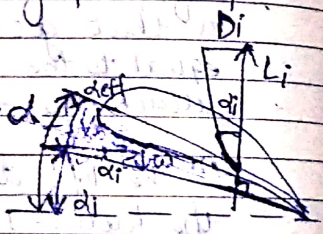
(iii) The induced drag per unit span:

$D_i' = L_i' \sin\alpha_i$

For small angle α,

$\sin\alpha_i = \alpha_i$

$\therefore D_i' = L_i'\alpha_i$



Total induced drag, $$D_i = \int_{-b/2}^{b/2} L'(y)\,\alpha_i(y)\,dy$$

$$\therefore D_i = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,\alpha_i(y)\,dy$$

∴ Induced drag coefficient:

$$C_{D_i} = \frac{D_i}{q_\infty S} = \frac{\rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,\alpha_i(y)\,dy}{\frac{1}{2}\rho_\infty V_\infty^2 S}$$

$$\therefore \boxed{C_{D_i} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\,\alpha_i(y)\,dy}$$

x——x

* Limitations of lifting-line theory:

The lifting line theory does not take into account the following:

(i) compressible flow.
(ii) viscous flow.
(iii) swept wings.
(iv) Low aspect ratio wings.
(v) Unsteady flows.

(Aspect ratio is the ratio of a wing's length to its mean chord).

x——x